ON THE COMPLETE CLASSIFICATION OF
EXTREMAL LOG ENRIQUES SURFACES

Keiji Oguiso and De-Qi Zhang

Abstract. We show that there are exactly, up to isomorphisms, seven rational extremal log Enriques surfaces Z and construct all of them; among them types D_{19} and A_{19} have been shown of certain uniqueness by M. Reid. We also prove that the (degree 3 or 2) canonical covering of each of these seven Z has either X_3 or X_4 as its minimal resolution. Here X_3 (resp. X_4) is the unique K3 surface with Picard number 20 and discriminant 3 (resp. 4), which are called the most algebraic K3 surfaces by Vinberg and have infinite automorphism groups (by Shioda-Inose and Vinberg).

INTRODUCTION

Throughout this paper, we work over the complex number field $\mathbb{C}$. A normal projective surface Z with at worst quotient singularities is called *a log Enriques surface* if the canonical Weil divisor K_Z is numerically equivalent to zero and if the irregularity dim $H^1(Z, \mathcal{O}_Z) = 0$ [Z1, (1.1)]. Note that a log Enriques surface is irrational if and only if it is a K3 or Enriques surface with at worst Du Val singular points, and also we can regard log Enriques surfaces as degenerations or generalizations of K3 and Enriques surfaces.

Rational log Enriques surfaces also appear as base spaces W of elliptically fibred Calabi-Yau threefolds $\Phi_{|D|} : X \to W$ with $D.c_2(X) = 0$ [O1]. On the other hand, a special case of [Al, Theorem 3.9] says that there are only finitely many deformation types of minimal resolutions of rational log Enriques surfaces. This also shows the sort of feasibility to classify them all.

Since the minimal partial resolution of the Du Val singular points of a log Enriques surface is again a log Enriques surface of the same canonical index (see below for the definition), we assume throughout this paper that a log Enriques surface has no Du Val singular points.

Let Z be a log Enriques surface and

$$I := I(Z) = \min\{n \in \mathbb{Z}_{>0} | \mathcal{O}_Z(nK_Z) \simeq \mathcal{O}_Z\}$$

the canonical index of Z. *The canonical covering* of Z is then defined as

$$\pi : \overline{S} := Spec_{\mathcal{O}_Z}(\oplus_{i=0}^{I-1} \mathcal{O}_Z(-iK_Z)) \to Z.$$

It follows from [K, Z1] that

    (1) $\overline{S}$ is either a projective K3 surface with at worst Du Val singularities or an abelian surface,

    (2) π is a finite, cyclic Galois cover of degree I and is étale over $Z - \text{Sing}Z$, and

(3) the Galois group $\mathrm{Gal}(\overline{S}/Z) \simeq \mathbb{Z}/I\mathbb{Z}$ acts faithfully on $H^0(\mathcal{O}_{\overline{S}}(K_{\overline{S}}))$, that is, there exists a generator g of $\mathrm{Gal}(\overline{S}/Z)$ such that $g^*\omega_{\overline{S}} = \zeta_I\omega_{\overline{S}}$, where $\zeta_I = \exp(2\pi i/I)$ and $\omega_{\overline{S}}$ is a nowhere vanishing regular 2-form on $\overline{S}$.

One interesting problem is to determine all possible canonical indices; in this aspect, [Bl] has shown that the canonical index is always less than or equal to 21 (see also [Z1,2]). On the other hand, in [Z1,2] for each prime number $p \leq 19$, we have constructed a rational log Enriques surface Z_p of index p, with the canonical covering $\pi : Y_p \to Z_p$, the Galois group $G = Gal(Y_p/Z_p)$ and the minimal resolution $X_p \to Y_p$, while in [OZ3] we have shown that for each $p = 13, 17, 19$ the pair (X_p, G) is unique up to isomorphisms.

Let $\nu : S \to \overline{S}$ be the minimal resolution of $\overline{S}$ and Δ_Z the exceptional divisor of ν. Then Δ_Z is a disconnected sum of divisors of Dynkin's type, $(\oplus A_\alpha) \oplus (\oplus D_\beta) \oplus (\oplus E_\gamma)$. Then, by abuse of language, we say that a log Enriques surface Z or the exceptional divisor Δ_Z is of type $(\oplus A_\alpha)\oplus(\oplus D_\beta)\oplus(\oplus E_\gamma)$. We define rank$\Delta_Z$ as the rank of the sublattice of the Néron Severi lattice $NS(S) = \mathrm{Pic}Z$ generated by the irreducible components of Δ_Z. Note that rankΔ_Z is the number $\sum \alpha + \sum \beta + \sum \gamma$ of the exceptional curves and satisfies

$$\mathrm{rank}\Delta_Z \leq \mathrm{rank}NS(S) - 1 \leq 19.$$

Our standpoint here is, as in previous [OZ1, 2], to consider rankΔ_Z as an invariant measuring how "big" $\mathrm{Sing}(Z)$ is.

Definition. *A rational log Enriques surface Z is said to be extremal if rank$\Delta_Z = 19$, the maximal possible value.*

Note that the minimal resolution S of the canonical cover of an extremal log Enriques surface is necessarily a singular K3 surface, that is, a smooth K3 surface with maximal possible Picard number 20, in the terminology of [SI]. Thus, it is very natural to ask whether we can show the uniqueness of each extremal type, up to isomorphisms [see Question 2 below].

In [OZ2], we have determined isomorphism classes of rational log Enriques surfaces of type D_{18} (one class only) or A_{18} (two classes), while in [OZ1] we gave an affirmative answer to the following question raised to the second author by I. Naruki and M. Reid when they saw the examples of rational log Enriques surfaces of type D_{19} or A_{19} in [Z1].

Question 1. *Are the rational log Enriques surfaces of type D_{19} and of type A_{19} unique respectively up to isomorphisms?*

This question is now naturally generalised to the following:

Question 2. *How about the extremal log Enriques surfaces?*

The main purpose of this paper is to give a complete answer to Question 2:

Main Theorem.
(1) [Restriction] Each extremal log Enriques surface has one of the following seven types:

D_{19}, $D_{16} \oplus A_3$, $D_{13} \oplus A_6$, $D_7 \oplus A_{12}$, $D_7 \oplus D_{12}$, $D_4 \oplus A_{15}$, *or* A_{19}.

(2) [Existence] Conversely, for each type Δ given in (1), there exists an extremal

(3) [Uniqueness] Extremal log Enriques surfaces are isomorphic if and only if their types are the same.

In particular, there exist exactly seven extremal log Enriques surfaces up to isomorphisms.

In section 1, we explicitly construct an extremal log Enriques surface of each type given in (1) via Shioda-Inose's pairs $(S_3, < g_3 >)$ and $(S_2, < g_2 >)$, that is, pairs of the singular K3 surfaces with two smallest discriminants 3 and 4 and their certain automorphism group of order 3 and 2 respectively. (See §1 for the detail.) As in [OZ1], the basic strategy of the proof here for the Main Theorem is to reduce problems of an extremal log Enriques surface to those of a singular K3 surfaces via the canonical covering and its minimal resolution, the so-called Godeaux approach. In section 2, we show the following proposition, which determines extremal log Enriques surfaces except for an ambiguity of the exceptional divisor Δ_Z of $S \to \overline{S}$, and is one of the cores of this paper:

Proposition (cf. Proposition (2.2)). *Let Z be an extremal log Enriques surface, $\overline{S} \to Z$ the canonical cover of Z and $S \to \overline{S}$ the minimal resolution of $\overline{S}$. Let $< g >$ be the automorphism group of S induced by the Galois group of $\overline{S} \to Z$. Then, the pair $(S, < g >)$ is isomorphic to either one of Shioda-Inose's pairs $(S_3, < g_3 >)$ or $(S_2, < g_2 >)$. In particular, the canonical index of an extremal log Enriques surface is either 3 or 2.*

The hardest part of this proposition is the determination of the canonical indices of extremal log Enriques surfaces. For this, we need some detailed analysis of the fixed locus $S^{<g>}$ based on Atiyah-Singer-Segal's holomorphic Lefschetz fixed point formula [AS1,2] and the usual topological Lefschetz fixed point formula (see eg. [U]). This analysis, which describes the fixed locus of an order 6 automorphism τ on a K3 surface T with $\tau^*\omega = \zeta_6\omega_T$, will be applicable to quite general cases. After proving $I = 2$ or 3, we apply the characterisation of Shioda-Inose's pairs $(S_3, < g_3 >)$ and $(S_2, < g_2 >)$ (Theorems (1.3) and (1.6)) to conclude Proposition(2.2).

In section 3, studying Δ as a sublattice of $NS(S)$, we show the Main Theorem (1). In section 4, we prove the Main Theorem (3) along the strategy given in (4.1).

NOTATION

For an automorphism group G and its element g of a smooth surface or a curve X, we set
$X^g := \{x \in X | g(x) = x\}$, the fixed locus of an element g of G,
$X^{[G]} := \cup_{g \in G - \{id\}} X^g = \{x \in X | g(x) = x \text{ for some } g \in G - \{id\}\}$. Note that X^g is a smooth algebraic set.
A curve C on a surface S is said to be g−stable (resp. g−fixed) if $g(C) = C$ (resp. $C \subset S^g$). We call $P \in S^g$ an isolated point if P is not contained in any g−fixed curves.

We denote by $\zeta_I = \exp(2\pi i/I)$, a specified primitive I−th root of unity.

Let Z be an extremal log Enriques surface. We set:

$I = I(Z)$ the canonical index of Z;

$\pi : \overline{S} \to Z$ the canonical cover of Z;

g the generator of $Gal(\overline{S}/Z)(\simeq \mathbb{Z}/I\mathbb{Z})$ such that $g^*\omega_{\overline{S}} = \zeta_I\omega_{\overline{S}}$;

$\nu : S \to \overline{S}$ the minimal resolution of $\overline{S}$;

$\Delta = \Delta_Z$ the exceptional divisor of ν;

$\Delta = \Delta_1 \coprod \cdots \coprod \Delta_r$ the decomposition of Δ into the connected components.
For the simplicity of notation, we also denote by the same letter g the induced action of g on S.

§1. CONSTRUCTION OF EXTREMAL LOG ENRIQUES SURFACES FROM SHIODA-INOSE'S PAIRS

First, we recall definitions and some properties of Shioda-Inose's pairs $(S_3, < g_3 >)$ and $(S_2, < g_2 >)$ from [SI] and [OZ1]. These pairs will play essential roles throughout this paper. Next we construct extremal log Enriques surfaces of all types shown in the Main Theorem (1). This will complete main Theorem (2), the existence part.

Definition (1.1) [OZ1, Example 1]. Let E_{ζ_3} be the elliptic curve of period ζ_3. Let $\overline{S_3} := E_{\zeta_3}^2/ < \mathrm{diag}(\zeta_3, \zeta_3^2) >$ be the quotient surface of the product $E_{\zeta_3}^2$ by $< \mathrm{diag}(\zeta_3, \zeta_3^2) >$ and $S_3 \to \overline{S_3}$ the minimal resolution of $\overline{S_3}$.

Let g_3 be the automorphism of S_3 (of order 3) induced by the action $\mathrm{diag}(\zeta_3, 1)$ on $E_{\zeta_3}^2$. We call the pair (S_3, g_3) or $(S_3, < g_3 >)$ Shioda-Inose's pair of discriminant 3.

Remark. It is shown in [SI] that S_3 is uniquely determined by the property that S_3 is a singular K3 surface whose transcendental lattice is of discriminant 3. This is the source of the name given in (1.1).

The next Proposition and Theorem are shown in [SI, OZ1].

Proposition (1.2). *Let (S_3, g_3) be the Shioda-Inose's pair of discriminant 3. Then,*

(1) *S_3 contains 24 rational curves: F_1, F_2, F_3 coming from $(E_{\zeta_3})^{\zeta_3} \times E_{\zeta_3}$, G_1, G_2, G_3 coming from $E_{\zeta_3} \times (E_{\zeta_3})^{\zeta_3}$, and E_{ij}, E'_{ij} $(1 \leq i, j \leq 3)$ the exceptional curves arising from the nine Du Val singular points (of Dynkin type A_2) of $\overline{S_3}$ (see [SI, Fig.2] or Figure 1 at the end of this paper for the configuration of these 24 curves),*

(2) *$g_3^*\omega_{S_3} = \zeta_3\omega_{S_3}$ and each of the 24 curves in (1) is g_3−stable,*

(3) *$S_3^{g_3} = (\coprod_{i=1}^3 F_i) \coprod (\coprod_{j=1}^3 G_j) \coprod (\coprod_{i,j=1}^3 \{P_{ij}\})$, where P_{ij} denote the point $E_{ij} \cap E'_{ij}$, and*

(4) *$g_3 \circ \varphi = \varphi \circ g_3$ for all $\varphi \in Aut(S_3)$.*

Theorem (1.3). *Let (S, g) be a pair of a smooth K3 surface and an automorphism g of S. Assume that (S, g) satisfies the following four conditions:*

(1) *$g^3 = id$,*

(2) *$g^*\omega_S = \zeta_3\omega_S$,*

(3) *S^g consists of only rational curves and isolated points, and*

Then $(S, g) = (S_3, g_3)$ up to equivariant isomorphisms. Moreover, S^g consists of exactly six rational curves and nine isolated points.

Definition (1.4) [OZ1, Example 2]. Let E_{ζ_4} be the elliptic curve of period ζ_4. Let $\overline{S_2} := E_{\zeta_4}^2 / < \operatorname{diag}(\zeta_4, \zeta_4^3) >$ be the quotient surface of the product $E_{\zeta_4}^2$ by $< \operatorname{diag}(\zeta_4, \zeta_4^3) >$ and $S_2 \to \overline{S_2}$ the minimal resolution of $\overline{S_2}$.

Let g_2 be the involution of S_2 induced by the action $\operatorname{diag}(-1, 1)$ on $E_{\zeta_4}^2$. We call the pair (S_2, g_2) Shioda-Inose's pair of discriminant 4.

Remark. It is also shown in [SI] that S_2 is uniquely determined by the property that S_2 is a singular K3 surface whose transcendental lattice is of discriminant 4.

Proposition(1.5) and Theorem(1.6) below are also shown in [SI, OZ1].

Proposition (1.5). *Let (S_2, g_2) be Shioda-Inose's pair of discriminant 4. Then,*

(1) S_2 *contains 24 rational curves:* F_1, F_2, F_3 *coming from* $(E_{\zeta_4})^{[<\zeta_4>]} \times E_{\zeta_4}$, G_1, G_2, G_3 *coming from* $E_{\zeta_4} \times (E_{\zeta_4})^{[<\zeta_4>]}$, $E_{11}' + H_{11} + E_{11}$, $E_{13}' + H_{13} + E_{13}$, $E_{31}' + H_{31} + E_{31}$, $E_{33}' + H_{33} + E_{33}$, *the exceptional curves arising from the four Du Val singular points (of Dynkin type A_3) of* $\overline{S_2}$, *and* E_{12}, E_{22}, E_{32} E_{21}', E_{22}', E_{23}', *the exceptional curves arising from the six Du Val singular points (of type A_1) of* $\overline{S_2}$ *(see [SI, Fig.3] or Figure 2 at the end of this paper for the configuration of these 24 curves),*

(2) $g_2^* \omega_{S_2} = -\omega_{S_2}$ *and each of the 24 curves in (1) is* g_2-*stable,*

(3) $S_2^{g_2} = (\coprod_{i=1}^3 F_i) \coprod (\coprod_{j=1}^3 G_j) \coprod (\coprod_{i,j=1}^3 H_{ij})$, *and*

(4) $g_2 \circ \varphi = \varphi \circ g_2$ *for all* $\varphi \in Aut(S_2)$.

Theorem (1.6). *Let (S, g) be a pair of a smooth K3 surface and an automorphism g of S. Assume that (S, g) satisfies the following four conditions:*

(1) $g^2 = id.$,

(2) $g^* \omega_S = -\omega_S$,

(3) S^g *consists of only rational curves, and*

(4) S^g *contains at least ten rational curves.*

Then $(S, g) = (S_2, g_2)$ up to equivariant isomorphisms. Moreover, S^g consists of exactly ten rational curves.

Now, using the notation in (1.1), (1.2), (1.4), (1.5) and tracing out Figures 1 and 2, we can easily construct an extremal log Enriques surface of each type given in the Main Theorem (1) as follows:

Proposition (1.7).

(1) *Let $\Delta(i)$ ($1 \le i \le 6$) be the divisors on S_3 defined by:*

$\Delta(1) = E_{11} + E_{21} + G_1 + E_{31} + E_{31}' + F_3 + E_{33}' + E_{33} + G_3 + E_{23} + E_{23}' + F_2 + E_{22}' + E_{22} + G_2 + E_{12} + E_{12}' + F_1 + E_{13}'$ *(of Dynkin type D_{19});*

$\Delta(2) = (E_{11}' + E_{12}' + F_1 + E_{13}' + E_{13} + G_3 + E_{23} + E_{23}' + F_2 + E_{22}' + E_{22} + G_2 + E_{32} + E_{32}' + F_3 + E_{33}') + (E_{21} + G_1 + E_{31})$ *(of Dynkin type $D_{16} \oplus A_3$);*

$\Delta(3) = (E_{12}' + E_{13}' + F_1 + E_{11}' + E_{11} + G_1 + E_{21} + E_{21}' + F_2 + E_{22}' + E_{22} + G_2 + E_{32}) + (E_{31}' + F_3 + E_{33}' + E_{33} + G_3 + E_{23})$ *(of Dynkin type $D_{13} \oplus A_6$);*

$\Delta(4) = (E_{11}' + E_{12}' + F_1 + E_{13}' + E_{13} + G_3 + E_{23}) + (E_{33}' + F_3 + E_{32}' + E_{32} +$

$\Delta(5) = (E'_{11} + E'_{12} + F_1 + E'_{13} + E_{13} + G_3 + E_{23}) + (E'_{33} + E'_{32} + F_3 + E'_{31} +$
$E_{31} + G_1 + E_{21} + E'_{21} + F_2 + E'_{22} + E_{22} + G_2)$ *(of Dynkin type $D_7 \oplus D_{12}$);*

$\Delta(6) = (E'_{11} + E'_{12} + E'_{13} + F_1) + (E_{33} + G_3 + E_{23} + E'_{23} + F_2 + E'_{22} + E_{22} +$
$G_2 + E_{32} + E'_{32} + F_3 + E'_{31} + E_{31} + G_1 + E_{21})$ *(of Dynkin type $D_4 \oplus A_{15}$).*

    Let $S_3 \to S(i)$ ($1 \le i \le 6$) be the contraction of $\Delta(i)$. Then the automorphism g_3 descends to automorphisms of $S(i)$, and the quotient surfaces $Z(i) := S(i)/ < g_3 >$ ($1 \le i \le 6$) are extremal log Enriques surfaces of type D_{19}, $D_{16} \oplus A_3$, $D_{13} \oplus A_6$, $D_7 \oplus A_{12}$, $D_7 \oplus D_{12}$ and $D_4 \oplus A_{15}$ respectively.

(2) *Let $\Delta(7)$ be the divisor on S_2 defined by*

$\Delta(7) = H_{31} + E'_{31} + F_3 + E'_{33} + H_{33} + E_{33} + G_3 + E_{13} + H_{13} + E'_{13} + F_1 +$
$E'_{11} + H_{11} + E_{11} + G_1 + E'_{21} + F_2 + E'_{22} + G_2$ *(of Dynkin type A_{19}).*

    Let $S_2 \to S(7)$ be the contraction of $\Delta(7)$. Then the automorphism g_2 descends to an automorphism of $S(7)$, and the quotient surface $Z(7) := S(7)/ < g_2 >$ is an extremal log Enriques surface of type A_{19}.

Proof. Since each connected component of $\mathrm{Supp}\Delta(i)$ ($1 \le i \le 6$) is g_3−stable, g_3 descends to its namesake on $S(i)$. Since in addition every 1-dimensional component of $S_3^{g_3}$ lies in $\mathrm{Supp}\,\Delta(i)$ and since no connected component of $\Delta(i)$ is disjoint from $S_3^{g_3}$, it follows that the quotient map $S(i) \to Z(i)$ has no ramification curves and that $Z(i)$ has no Du Val singular points. Thus, $Z(i)$ is a log Enriques surface whose canonical cover is equal to the quotient map $S(i) \to Z(i)$. This implies the assertion (1). The verification of (2) is also similar. $\square$

§2. GLOBAL CANONICAL COVER OF AN EXTREMAL LOG ENRIQUES SURFACE

Note (2.1). In this section, we let Z be an extremal log Enriques surface of index I, and we shall use the notation in the Introduction.

    The goal of this section is to show the following:

Proposition (2.2).
    (1) *The canonical index I is either 2 or 3.*
    (2) *In the case where $I = 2$, (S, g) is isomorphic to Shioda-Inose's pair (S_2, g_2) of discriminant 4 and Z is isomorphic to the extremal log Enriques surface $Z(7)$ defined in $(1.7)(2)$.*
    (3) *In the case where $I = 3$, (S, g) is isomorphic to Shioda-Inose's pair (S_3, g_3) of discriminant 3 and the type of Z is either D_{19}, $D_{3l+1} \oplus D_{3m}$ or $D_{3l+1} \oplus A_{3m}$, where l and m are positive integers with $l + m = 6$.*

    This Proposition will immediately follow from Lemmas (2.4), (2.8), (2.9), (2.11) and (2.13) below. First we remark some easy facts.

Lemma (2.3).
    (1) *Every curve in $S^{[<g>]}$ is contained in Δ. In particular, $S^{[<g>]}$ consists of smooth rational curves and finite isolated points.*
    (2) *Δ is g−stable, that is, $g(\Delta) = \Delta$.*

Proof. Since $K_Z \equiv 0$, the quotient map $\overline{S} \to Z$ is unramified in codimension one. This implies the assertion (1). The assertion (2) is clear. $\square$

Lemma (2.4). *I is either 2, 3, 4, or 6.*

Proof. Since S is a singular K3 surface, we know that $\operatorname{rank} T_S = 2$, where T_S denotes the transcendental lattice of S. Since $g^*\omega_S = \zeta_I \omega_S$ and $\omega_S \in T_S \otimes \mathbb{C}$, the action g^* on T_S has an eigen value ζ_I. Thus, $\varphi(I) \le 2$, where φ is the Euler function. This implies the result. $\square$

We quote here the next two easy but useful Lemmas from [OZ1].

Lemma (2.5) ([OZ1, Lemma 3.2]). *Let T be a smooth K3 surface and τ an involution of T such that $\tau^*\omega_T = -\omega_T$.*

   (1) *Let C_1 and C_2 be two $g-$stable smooth rational curves on T with $C_1 \cdot C_2 = 1$. Then, exactly one of C_i is $\tau-$fixed.*

   (2) *Let C be a $\tau-$stable but not $\tau-$fixed smooth rational curve on T. Set $C \cap T^\tau = \{P_1, P_2\}$. Then, for each $i = 1, 2$, there exists a $\tau-$fixed curve D_i passing through P_i.*

Lemma (2.6) ([OZ1, Lemma 2.2, Proof of Lemma 2.3]). *Let T be a smooth K3 surface with an automorphism τ of T. Assume that τ is of order 3 and that $\tau^*\omega_T = \zeta_3 \omega_T$.*

   (1) *Let $C_1 + C_2 + C_3$ be a linear chain of smooth rational curves on T. Assume that each C_i is $\tau-$stable. Then, exactly one of C_i is $\tau-$ fixed.*

   (2) *Let C be a $\tau-$stable but not $\tau-$fixed smooth rational curve on T. Then, there exists a $\tau-$fixed curve D on T with $C \cdot D = 1$.*

   (3) *Assume that T^τ consists of rational curves and isolated points. Let N (resp. M) be the number of rational curves (resp. isolated points) in T^τ. Then $M - N = 3$.*

We return to our initial situation (2.1).

Lemma (2.7). *Assume that $I = 2$. Then we have:*

   (1) *Each connected component Δ_i of Δ is $g-$stable.*

   (2) *Δ_i is of type A_{2n_i+1} ($n_i \in \mathbb{Z}_{\ge 0}$). (See [Z1, Lemma 3.1].)*

   (3) *Each irreducible component of Δ_i is $g-$stable.*

   (4) *Let $\Delta_i = C_1 + C_2 + \cdots + C_{2n_i+1}$ be the irreducible decomposition of Δ_i such that the dual graph of Δ_i is $C_1 - C_2 - \cdots - C_{2n_i+1}$. Then, C_j is $g-$fixed if and only if $j \equiv 1 (mod 2)$.*

Remark. This Lemma requires our assumption that a log Enriques surface has no Du Val singular points.

Proof. We proceed the proof dividing into four steps.

Step 1. *Each Δ_i is $g-$stable.*

Proof. This follows from our assumption that Z has no Du Val singular points. $\square$

Step 2. *Δ_i is of type A_m for certain integer m.*

Proof. Assume the contrary that Δ_i is not of type A_m. Then, by the classification of Dynkin diagram, there exists a unique rational curve C in Δ_i which meets exactly three rational curves in Δ_i, say, D_1, D_2, and D_3. Note that at least one of D_j, say

$g(C) = C$ and $g(\{D_1, D_2, D_3\}) = \{D_1, D_2, D_3\}$. We shall derive a contradiction dividing into the two cases:

Case 1. $g|C = id$ and *Case 2.* $g|C \neq id$.

Case 1. In this case, D_1 is $g-$stable but not $g-$fixed $((2.5)(1))$ and D_1^g consists of two points. Since one of these two points is not in C, there exists a $g-$fixed curve $E(\neq C)$ which meets D_1 $((2.5)(2))$. This implies $E \subset \Delta_i$ $((2.3)(1))$, a contradiction to the choice of D_1.

Case 2. There exists exactly one D_j with $g(D_j) = D_j$. Thus, C^g contains a point Q which does not lie in $D_1 \cup D_2 \cup D_3$. Then, there exists a $g-$fixed curve $E(\neq D_1, D_2, D_3)$ passing through Q $((2.5)(2))$. This implies $E \subset \Delta_i$ $((2.3)(1))$, a contradiction to the choice of C. $\square$

Step 3. *Write* $\Delta_i = \sum_{j=1}^m C_j$ *with* $C_j.C_{j+1} = 1$ $(1 \leq j \leq m-1)$. *Then each irreducible component* C_j *is* $g-$*stable.*

Proof. Assume the contrary that $g(C_j) \neq C_j$ for some j. Then, $g(C_j) = C_{m+1-j}$ for all j, because $\mathrm{Aut}_{\mathrm{graph}}(A_m) \simeq \mathbb{Z}/2$. We shall drive a contradiction dividing into the two cases:

Case 1. $m \equiv 0 \pmod 2$, *Case 2.* $m \equiv 1 \pmod 2$.

Case 1. Since $g(C_{m/2} \cap C_{m/2+1}) = C_{m/2} \cap C_{m/2+1}$, there exists a $g-$fixed curve $E(\neq C_{m/2}, C_{m/2+1})$ passing through the point $C_{m/2} \cap C_{m/2+1}$. Then $E \subset \Delta_i$ $((2.3)(1))$, a contradiction.

Case 2. Since $C_{(m+1)/2}$ is $g-$stable but not $g-$fixed, there exists a $g-$fixed curve E meeting $C_{(m+1)/2}$. Then $E \subset \Delta_i$, a contradiction. $\square$

Step 4. $m \equiv 1 (mod 2)$; *and* $g|C_j = id$ *if* $j \equiv 1 (mod 2)$, *while* $g|C_j$ *is an involution if* $j \equiv 0 (mod 2)$.

Proof. It follows from Step 3, $(2.5)(1)$ and $(2.3)(1)$ that both C_1 and C_m are $g-$fixed. Now the result readily follows from $(2.5)(1)$. $\square$

This completes the proof of (2.7). $\square$

Lemma (2.8). *Assume that* $I = 2$. *Then* (S, g) *is isomorphic to Shioda-Inose's pair* (S_2, g_2) *of discriminant* 4 *and* Z *is isomorphic to the extremal log Enriques surface* $Z(7)$ *defined in* $(1.7)(2)$.

Proof. Let N be the number of $g-$fixed curves on S. Recall that Δ_i contains just $(n_i + 1)$ $g-$fixed curves (2.7) and that every $g-$fixed curve is contained in $\Delta = \coprod_{i=1}^r \Delta_i$ $(2.3)(1)$. Thus, we get

$$(1) \qquad N = \sum_{i=1}^r (n_i + 1) = r + \sum_{i=1}^r n_i.$$

On the other hand, since Z is extremal, we have

$$(2) \qquad 19 = \mathrm{rank}\Delta = \sum_{i=1}^r (2n_i + 1) = r + 2(\sum_{i=1}^r n_i).$$

Combining these two equalities, we get

$$(3) \qquad N \quad \quad + (19 \quad \quad)/2 \quad (19 +)/2 > 10$$

Now we may apply (1.6) to get $(S, g) \simeq (S_2, g_2)$ and $N = 10$. This implies that $r = 1$ and that Δ is of type A_{19}. In other words, Z is the most extremal log Enriques surface of type A_{19}. Now the result follows from [OZ1, main Theorem 2]. $\square$

Lemma (2.9). $I \neq 4$.

Proof. Assume the contrary that $I = 4$. Then $h := g^2$ is an involution of S with properties that $h^*\omega_S = -\omega_S$ and $S^h \subset \Delta$.

Claim. *Each Δ_i is $h-$stable.*

Proof. Assume the contrary that $h(\Delta_i) \neq \Delta_i$ for some i. Then Δ_i, $g(\Delta_i)$, $h(\Delta_i)$, and $g^3(\Delta_i)$ are mutually different connected components of Δ. Thus, $Z = \overline{S}/ < g >$ has a Du Val singular point $\pi \circ \nu(\Delta_i)$, a contradiction. $\square$

By virtue of this Claim, we may repeat the same argument as in Steps 2-4 in (2.7) and (2.8) for the pair (S, h) to show that Δ is of type A_{19}. This implies that Z is the most extremal log Enriques surface of type A_{19}. However, then $I = 2$ by [OZ1, main Theorem 2], a contradiction. $\square$

Lemma (2.10). *Assume that $I = 3$.*

(1) *Each Δ_i is $g-$stable. Moreover, each irreducible component of Δ_i is also $g-$stable.*

(2) *Δ_i is either of type A_{n_i} or of type D_{m_i} with $m_i \not\equiv 2(mod 3)$. (See [Z1, Prop. 6.1].)*

(3) *All possible configurations of Δ_i are given as follows, where we denote by f (resp. s) $g-$fixed (resp. $g-$stable but not $g-$fixed) irreducible components:*

$$f \ (\text{type } A_1), \quad f - s \ (\text{type } A_2), \quad s - f - s \ (\text{type } A_3)$$

$$s - f - s - s - f - s - s - \cdots - f - s - s - f - s \ (\text{type } A_{3p})$$

$$f - s - s - f - s - s - \cdots - f - s - s - f - s \ (\text{type } A_{3p-1})$$

$$f - s - s - f - s - s - \cdots - f - s - s - f \ (\text{type } A_{3p-2})$$

$$\begin{array}{c} s \\ | \\ f - s - s - f - s - s - \cdots - f - s - s - f - s \ (\text{type } D_{3q+1}) \\ | \\ s \end{array}$$

$$s$$
$$|$$

$$f - s - s - f - s - s - \cdots - f - s - s - f \quad (\text{type } D_{3q})$$

$$|$$
$$s$$

In particular, the pair of
(the number of $g-$fixed curves, the number of $g-$fixed isolated points)
for each Δ_i is as follows:
  $(p, p+1)$ if of type A_{3p}; (p, p) if of type A_{3p-1}; $(p, p-1)$ if of type A_{3p-2};
  $(q, q+2)$ if of type D_{3q+1}; $(q, q+1)$ if of type D_{3q}.

Proof. Making use of (2.3) and (2.6) (instead of (2.3) and (2.5)) and tracing out Dynkin diagrams, we can prove (2.10) in the same manner as in (2.7). Details will be left to the readers. $\square$

Lemma (2.11). *Assume that $I = 3$. Then, (S, g) is isomorphic to Shioda-Inose's pair (S_3, g_3) of discriminant 3 and the type of Z is either D_{19}, $D_{3l+1} \oplus D_{3m}$ or $D_{3l+1} \oplus A_{3m}$, where l and m are positive integers with $l + m = 6$.*

Proof. Let N (resp. M) be the number of $g-$fixed curves (resp. $g-$fixed isolated points) on S. Then by (2.6)(3), we have

$$(1) \qquad\qquad\qquad\qquad M - N = 3.$$

On the other hand, we know by (2.10) that Δ is a disjoint sum of $a + b + c + d + e$ divisors whose types are:
  $D_{3l_1+1}, \cdots, D_{3l_a+1}, D_{3m_1}, \cdots, D_{3m_b},$
  $A_{3p_1}, \cdots, A_{3p_c}, A_{3q_1-1}, \cdots, A_{3q_d-1},$ and $A_{3r_1-2}, \cdots, A_{3r_e-2},$
  where a, b, c, d, and e are certain non-negative integers.
  Then using (2.3)(1) and (2.10)(3), we calculate

$$(2) \qquad\qquad N = \sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j + \sum_{k=1}^{c} p_k + \sum_{l=1}^{d} q_l + \sum_{m=1}^{e} r_m$$

and
$$M \geq \sum_{i=1}^{a}(l_i + 2) + \sum_{j=1}^{b}(m_j + 1) + \sum_{k=1}^{c}(p_k + 1) + \sum_{l=1}^{d} q_l + \sum_{m=1}^{e}(r_m - 1)$$

$$(3) \qquad = \sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j + \sum_{k=1}^{c} p_k + \sum_{l=1}^{d} q_l + \sum_{m=1}^{e} r_m + (2a + b + c - e).$$

Substituting (2) and (3) into (1), we get

$$(4) \qquad\qquad\qquad 3 = M - N \geq 2a + b + c - e.$$

Since Z is an extremal log Enriques surface, we calculate
$$19 = \sum_{i=1}^{a}(3l_i + 1) + \sum_{j=1}^{b} 3m_j + \sum_{k=1}^{c} 3p_k + \sum_{l=1}^{d}(3q_l - 1) + \sum_{m=1}^{e}(3r_m - 2)$$

where we use (2) to get the last equality. Thus,

$$(5) \qquad N = \frac{19 + (2e + d - a)}{3}.$$

Suppose that $2e + d - a \leq -2$. Then $a \geq 2e + d + 2$. Substituting this into (4), we get

$$3 \geq 2a + b + c - e \geq 2(2e + d + 2) + b + c - e = 4 + b + c + 2d + 3e \geq 4,$$

a contradiction. Thus $2e + d - a \geq -1$. Substituting this into (5), we get $N \geq (19 - 1)/3 = 6$. Now we may apply (1.3) to get $(S, g) \simeq (S_3, g_3)$ and then $N = 6$. Combining this equality with (5), we get $2e + d - a = -1$, that is, $a = 2e + d + 1$. Substituting this into (4), we calculate

$$3 \geq 2(2e + d + 1) + b + c - e = 2 + 2d + b + c + 3e,$$

that is,

$$1 \geq 2d + 3e + b + c.$$

¿From this, we can easily see that $d = e = 0$, $a = 2e + d + 1 = 1$ and $b + c \leq 1$. Combining these formula with $\mathrm{rank}\Delta = 19$, we readily see that Δ is either one of the following types: D_{19}, $D_{3l+1} \oplus D_{3m}$, or $D_{3l+1} \oplus A_{3m}$ ($l + m = 6$). This is nothing but the last half assertion of (2.11). $\square$

It only remains to show $I \neq 6$. For this we need the following:

Proposition (2.12). *Let T be a smooth K3 surface and τ an automorphism of T. Assume that*

(1) *τ is of order 6 and $\tau^*\omega_T = \zeta_6\omega_T$, and that*
(2) *$T^{[<\tau>]}$ consists only of isolated points and smooth rational curves.*

Then $T^\tau (= T^{\tau^5})$, $T^{\tau^2}(= T^{\tau^4})$, and T^{τ^3} are described as follows:

$$T^\tau = (\coprod_{i=1}^{2(c+1)} \{P_i\}) \coprod (\coprod_{i=1}^{2(c+1)} \{Q_i\}) \coprod (\coprod_{j=1}^{c} C_j),$$

$$T^{\tau^2} = (\coprod_{i=1}^{2(c+1)} \{P_i\}) \coprod (\coprod_{k=1}^{2(p+1)} \{P'_k\}) \coprod (\coprod_{j=1}^{c} C_j) \coprod (\coprod_{l=1}^{c+1} D_l) \coprod (\coprod_{m=1}^{2p} F_m),$$

$$T^{\tau^3} = (\coprod_{j=1}^{c} C_j) \coprod (\coprod_{i=1}^{2(c+1)} E_i) \coprod (\coprod_{n=1}^{3q} G_n),$$

where c, p, and q are non-negative integers with $c + p + q \leq 2$, P_*, Q_*, and P'_* are isolated points, and C_*, D_*, E_*, F_*, G_* are smooth rational curves. Moreover, each of D_* and E_* is τ-stable, while τ acts on each set $\{F_{2i-1}, F_{2i}\}$ as an involution and on $\{G_{3j-2}, G_{3j-1}, G_{3j}\}$ of order 3.

Proof. Our proof is based on the holomorphic Lefschetz fixed point formula [AS1, 2], the topological Lefschetz fixed point formula [U], and local coordinate calculation.

Step 1.

$$T^{\tau} = \{P_1\} \coprod \cdots \coprod \{P_{2l}\} \coprod \{Q_1\} \coprod \cdots \coprod \{Q_{2l}\} \coprod C_1 \coprod \cdots \coprod C_c,$$

$$T^{\tau^2} = \{P_1\} \coprod \cdots \coprod \{P_{2l}\} \coprod \{P_1'\} \coprod \cdots \coprod \{P_{k'}'\}$$

$$\coprod C_1 \coprod \cdots \coprod C_c \coprod D_1 \coprod \cdots \coprod D_l \coprod F_1 \coprod \cdots \coprod F_{p'},$$

$$T^{\tau^3} = C_1 \coprod \cdots \coprod C_c \coprod E_1 \coprod \cdots \coprod E_{2l} \coprod G_1 \coprod \cdots \coprod G_{q'},$$

where l, c, p', k and q' are non-negative integers and C_*, D_*, F_*, E_*, and G_* are smooth rational curves. Moreover $Q_{2i-1}, Q_{2i} \in D_i$, $P_j, Q_j \in E_j$, and each of D_i and E_j is $\tau-$stable.

Proof. Suppose that P is an isolated point of T^{τ}. Since $\tau^* \omega_T = \zeta_6 \omega_T$, there exist local coordinates (x_P, y_P) around P such that either

(1) $\tau^*(x_P, y_P) = (\zeta_6^2 x_P, \zeta_6^5 y_P)$ or
(2) $\tau^*(x_P, y_P) = (\zeta_6^3 x_P, \zeta_6^4 y_P)$.

Denote by $P_1, \cdots, P_a (\in T^{\tau})$ (resp. by $Q_1, \cdots, Q_b (\in T^{\tau})$) the points of type (1) (resp. of type (2)). Then we write $T^{\tau} = \{P_1\} \coprod \cdots \coprod \{P_a\} \coprod \{Q_1\} \coprod \cdots \coprod \{Q_b\} \coprod C_1 \coprod \cdots \coprod C_c,$ where C_α are smooth rational curves. Let R be a point in C_α. Then there exist local coordinates (x_R, y_R) around R such that $\tau^*(x_R, y_R) = (x_R, \zeta_6 y_R)$. Note that $C_\alpha = (y_R = 0)$ around R.

Let P be a point in $\{P_1, ..., P_a\}$. Since $(\tau^*)^2(x_P, y_P) = (\zeta_6^4 x_P, \zeta_6^4 y_P)$ by (1), P is an isolated τ^2-fixed point.

Let Q be a point in $\{Q_1, ..., Q_b\}$. Since $(\tau^*)^2(x_Q, y_Q) = (x_Q, -y_Q)$ by (2), there exists a unique smooth rational curve, say D, such that $Q \in D \subset T^{\tau^2}$. Note that $D = (y_Q = 0)$ around Q. In particular, D is $\tau-$stable and $\tau|D$ is an involution on D. Thus τ has another $\tau-$fixed point Q' on D around which $(\tau|D)^* = (\zeta_6^4)^{-1}$. Since C_α and D are disjoint (by the smoothness of T^{τ^2}), this point Q' is also isolated in T^{τ} and in fact contained in $\{Q_1, ..., Q_b\}$. Now setting $\{D_j (1 \leq j \leq l) | D_j \subset T^{\tau^2}, D_j \cap \{Q_1, ..., Q_b\} \neq \phi\}$, and using the smoothness of T^{τ^2}, we can adjust the numbering of Q_β $(1 \leq \beta \leq b)$ as $Q_1, Q_2 \in D_1, ..., Q_{b-1}, Q_b \in D_l$. In particular, $b = 2l$.

Next we examine T^{τ^3}. Again, let P (resp. Q) be a point in $\{P_1, ..., P_a\}$ (resp. in $\{Q_1, ..., Q_{2l}\}$). Since $(\tau^*)^3(x_P, y_P) = (x_P, -y_P)$ around P, there exists a unique smooth rational curve E' such that $P \in E' \subset T^{\tau^3}$ (and that $E' = (y_P = 0)$ around P). Similarly, there exists a unique smooth rational curve E'' such that $Q \in E'' \subset T^{\tau^3}$ (and that $E'' = (x_Q = 0)$ around Q). Using this description, we easily see that both E' and E'' are $\tau-$stable and that $\tau|E'$ is a multiplication by ζ_6^2 around P and $\tau|E''$ is a multiplication by ζ_6^4 around Q. Note also that $|(E')^{\tau}| = 2$ and $|(E'')^{\tau}| = 2$.

Let E_i $(1 \leq i \leq m)$ be the τ^3-fixed curves which contains at least one point in $\{P_\alpha, Q_\beta | 1 \leq \alpha \leq a, 1 \leq \beta \leq 2l\}$. By the smoothness of T^{τ^3}, each E_i coincides with some E' or E'' found in the above process. In particular, each E_i is $\tau-$stable. Then, using again the smoothness of T^{τ^3} and the description of T^{τ}, and regarding the two points E^{τ} as 0 and ∞ of $E_i (\simeq \mathbb{P}^1)$, we see that there exist bijections $\varphi : \{1, ..., m\} \to \{1, ..., a\}$ and $\psi : \{1, ..., m\} \to \{1, ..., 2l\}$ such that $E_i^{\tau} = \{P_{\varphi(i)}, Q_{\psi(i)}\}$. Thus $m = a = 2l$. Then renumbering E_* and P_*, we have $P_i, Q_i \in E_i$ for all i with $1 \leq i \leq 2l$. Since T^{τ^3} contains no isolated points, we can now easily get the description of T^{τ^3} in Step 1. Now we get the desired description of $T^{\tau}, T^{\tau^2}, T^{\tau^3}$. $\square$

Step 2. $l = c+1$, $p' = 2p$, $q' = 3q$ and $k = 2(p+1)$ *for some non-negative integers* p *and* q, *where* l, p', q' *and* k *are integers found in Step 1.*

Proof. We apply the holomorphic Lefschetz fixed point formula [AS1,2] for (T, τ):

$$L(\tau) := \sum(-1)^i \mathrm{tr}(\tau^* | H^i(T, \mathcal{O}_T)) = \sum_{j=1}^{2l} a(P_j) + \sum_{j=1}^{2l} a(Q_j) + \sum_{i=1}^{c} b(C_i).$$

We calculate both sides and compare them.

Using the Serre duality, we get from the first equality that

$$(1) \qquad\qquad L(\tau) = 1 + \zeta_6^{-1} = \frac{3 - \sqrt{-3}}{2}.$$

By the definition of $a(*)$ aa in [AS1,2] and the local description of τ-action given in Step 1, we calculate

$$a(P_j) := \frac{1}{\det(1 - \tau^* | T_{P_j})} = \frac{1}{(1 - \zeta_6^2)(1 - \zeta_6^5)} = \frac{3 - \sqrt{-3}}{6},$$

$$a(Q_j) = \frac{1}{(1 - \zeta_6^3)(1 - \zeta_6^4)} = \frac{3 - \sqrt{-3}}{12},$$

$$b(C_i) := \frac{1 - g(C_i)}{1 - \zeta_6} - \frac{\zeta_6}{(1 - \zeta_6)^2} \cdot (C_i^2) = \frac{1}{1 - \zeta_6} - \frac{\zeta_6}{(1 - \zeta_6)^2} \cdot (-2) = \frac{-(3 - \sqrt{-3})}{2}.$$

Using the above formula for $L(\sigma)$ in terms of $a(*), b(*)$, we obtain:

$$(2) \qquad\qquad L(\tau) = \frac{3 - \sqrt{-3}}{6} \cdot 2l + \frac{3 - \sqrt{-3}}{12} \cdot 2l - \frac{(3 - \sqrt{-3})}{2} \cdot c.$$

Combining (1) and (2), we readily see that $l = c + 1$. Thus,

$$T^{\tau^2} = \{P_1\} \coprod \cdots \coprod \{P_{2(c+1)}\} \coprod \{P_1'\} \coprod \cdots \coprod \{P_{k'}'\}$$
$$\coprod C_1 \coprod \cdots \coprod C_c \coprod D_1 \coprod \cdots \coprod D_{(c+1)} \coprod F_1 \coprod \cdots \coprod F_{p'}.$$

Using this description and the smoothness of T^{τ^2}, we easily see that τ acts on both $\{F_1, ..., F_{p'}\}$ and $\{P_1', ..., P_{k'}'\}$ as fixed point free involutions. Thus, $p' = 2p$ and $k' = 2k$ for some integers p and k.

Next, we shall find a relation between k and p. Applying (2.6) to the pair (T, τ^2) where $\mathrm{ord}(\tau^2) = 3$, we obtain $\#(\tau^2 - \text{isolated points}) - \#(\tau^2 - \text{fixed curves}) = 3$, that is,

$$2(c + 1) + 2k - (c + (c + 1) + 2p) = 3.$$

This implies $k = p + 1$. Using the description of T^{τ^3} and applying the same argument as before for the set $\{G_1, G_2, \ldots, G_{q'}\}$ (instead of $\{F_1, F_2, \ldots, F_{p'}\}$), we can readily see that τ induces a fixed point free automorphism of order 3 on the set $\{G_1, ..., G_{q'}\}$. Thus, $q' = 3q$ for some integer q. This completes Step 2. $\square$

Now we only remain to show the inequality $c + p + q \leq 2$. Let us consider the action τ^* on $H^2(T, \mathbb{Q})$. Since $(\tau^*)^6 = id.$ and $\tau^* \omega_T = \zeta_6 \omega_T$, the pairs of (the eigenvalue of $\tau^* | H^2(T, \mathbb{Q})$, its multiplicity) are written as

$$(1, \alpha), (-1, \beta), (\zeta_3, \gamma), (\overline{\zeta_3}, \gamma), (\zeta_6, 1 + \delta), (\overline{\zeta_6}, 1 + \delta),$$

where α, β, γ and δ are certain non-negative integers. Now the required inequality

Step 3. *With the above notation,*
$\alpha = 5c + 2p + q + 6$,
$\beta = -c + 2p - q + 4$,
$\gamma = -c - p + q + 3$, *and*
$\delta = -c - p - q + 2$.

Proof. Since $\dim H^2(T, \mathbb{Q}) = 22$, we have

(1) $$\alpha + \beta + 2\gamma + 2\delta = 20.$$

In order to obtain other relations, we make use of the topological Lefschetz fixed point formula [U]:

(*) $$\chi_{top}(T^{\tau^j}) = \sum_{i=0}^{4} (-1)^i \mathrm{tr}((\tau^*)^j | H^i(S, \mathbb{Q})).$$

Using $T^\tau = \{P_1\} \coprod \cdots \coprod \{P_{2(c+1)}\} \coprod \{Q_1\} \coprod \cdots \coprod \{Q_{2(c+1)}\} \coprod C_1 \coprod \cdots \coprod C_c$ and applying (*) with $j = 1$, we get $4(c+1) + 2c = 2 + \alpha - \beta - \gamma + \delta + 1$. This gives

(2) $$\alpha - \beta - \gamma + \delta = 6c + 1.$$

Next using $T^{\tau^2} = \{P_1\} \coprod \cdots \coprod \{P_{2(c+1)}\} \coprod \{P_1'\} \coprod \cdots \coprod \{P_{2(p+1)}'\} \coprod C_1 \coprod \cdots \coprod C_c$ $\coprod D_1 \coprod \cdots \coprod D_{c+1} \coprod F_1 \coprod \cdots \coprod F_{2p}$ and applying (*) with $j = 2$, we get $2(c+1) + 2(p+1) + 2c + 2(c+1) + 2 \cdot 2p = 2 + (\alpha + \beta) - (\gamma + \delta + 1)$. This gives

(3) $$\alpha + \beta - \gamma - \delta = 6c + 6p + 5.$$

Finally using $T^{\tau^3} = C_1 \coprod \cdots \coprod C_c \coprod E_1 \coprod \cdots \coprod E_{2(c+1)} \coprod G_1 \coprod \cdots \coprod G_{3q}$ and applying (*) for τ^3, we get $2c + 2 \cdot 2(c+1) + 2 \cdot 3q = 2 + (\alpha + 2\gamma) - (\beta + 2(\delta + 1))$. This implies

(4) $$\alpha - \beta + 2\gamma - 2\delta = 6c + 6q + 4.$$

Now solving the equations (1) - (4) for α, β, γ, δ, we get the result. $\square$

This completes the proof of (2.12). $\square$

Returning back to our intial setting (2.1), we prove the following:

Lemma (2.13). $I \neq 6$.

Proof. Assume that $I = 6$. Then applying (2.12) for (S, g), we see that there are non-negative integers c, p, q such that $c + p + q \leq 2$ and that the irreducible decompositions of the 1-dimensional locus of S^g, S^{g^2} and S^{g^3} are written as follows respectively:

$C_1 \coprod \cdots \coprod C_c$;
$C_1 \coprod \cdots \coprod C_c \coprod D_1 \coprod \cdots \coprod D_{c+1} \coprod F_1 \coprod \cdots \coprod F_{2p}$;
$C_1 \coprod \cdots \coprod C_c \coprod E_1 \coprod \cdots \coprod E_{2(c+1)} \coprod G_1 \coprod \cdots \coprod G_{3q}$,
where C_*, D_*, and E_* are g−stable while the other F_* and G_* are not g−stable. Note also by (2.3)(1) that these curves C_*, D_*, E_* are all contained in Δ.

Let us consider the connected components Δ_i of Δ. Since $Z = \overline{S}/ <g>$ has no Du Val singular points, each Δ_i satisfies either

(1) g^3−stable or

Let Δ_i $(1 \leq i \leq n)$ be of type (1) and Δ_j $(n+1 \leq j \leq n+m = r)$ of type (2). Since g^3 is of order 2 and $(g^3)^*\omega_S = -\omega_S$, it follows from the argument in (2.7) (Steps 2-4) that each Δ_i $(1 \leq i \leq n)$ is of the Dynkin type $A_{2\alpha_i+1}$ and contains exactly $(\alpha_i + 1)$ g^3−fixed curves. On the other hand, the above description of S^{g^3} shows that the number of all the g^3−fixed curves is just $3(c+q)+2$. Thus,

$$(1) \qquad \sum_{i=1}^{n} \mathrm{rank}\Delta_i = \sum_{i=1}^{n}(2(\alpha_i+1)-1) = 2\sum_{i=1}^{n}(\alpha_i+1) - n = 6(c+q)+4-n.$$

Let us consider the connected components Δ_j of type (2). Since g^2 is of order 3 and $(g^2)^*\omega_S = \zeta_3\omega_S$, it follows from the argument in (2.10) that each Δ_j is of Dynkin type A_* or D_* and contains at least one g^2−fixed curve. Moreover, only F_* are the g^2−fixed curves in Δ_j, because C_* and D_* are g−stable so they are in Δ_i $(1 \leq i \leq n)$. Thus,

$$(2) \qquad\qquad\qquad\qquad\qquad m \leq 2p,$$

and $n \geq 1$ (because there is at least one D_*) and $\mathrm{rank}\Delta_j \leq 3 \cdot |\{F_*|F_* \subset \Delta_j\}| + 1$ (2.10(3)). Thus,

$$(3) \qquad\qquad\qquad \sum_{j=n+1}^{n+m} \mathrm{rank}\Delta_j \leq 3 \cdot 2p + m = 6p + m.$$

Combining (1), (2) and (3) with $19 = \sum_{i=1}^{n} \mathrm{rank}\Delta_i + \sum_{j=n+1}^{n+m} \mathrm{rank}\Delta_j$ and $c+p+q \leq 2$, we get

$$(4) \quad 19 \leq 6(c+q)+4-n+6p+m = 6(c+p+q)+4+m-n \leq 6\cdot2+4+2p-n \leq 19.$$

Thus the all inequalities in (4) must be equalities. This implies $n = 1$, $p = 2$, $m = 2p = 4$, and $c = q = 0$. Combining these equalities with rank $\Delta = 19$, we readily see that Δ is of type $A_3 \oplus D_4^{\oplus 4}$. Then using (2.10), we see that Δ contains $2 + 4 \cdot 3 = 14$ isolated g^2−fixed points and that g^2 has exactly 5 fixed curves. Thus, $M \geq 14$ and $N = 5$, where M is the number of the isolated g^2−fixed points and N is that of the g^2−fixed curves on S. However this contradicts the equality $M - N = 3$ ((2.6)(3)). This completes the proof. $\square$

§3. Types of extremal log Enriques surfaces

The goal of this section is to finish the proof of the Main Theorem (1).

Let Z be an extremal log Enriques surface of index I and we shall use the notation in the Introduction. By (2.11), we already know that Δ is either one of the following types: A_{19}, D_{19}, $D_{16} \oplus A_3$, $D_{13} \oplus A_6$, $D_{10} \oplus A_9$, $D_7 \oplus A_{12}$, $D_4 \oplus A_{15}$, $D_{13} \oplus D_6$, $D_{10} \oplus D_9$, $D_7 \oplus D_{12}$, $D_4 \oplus D_{15}$.

Thus, in order to get the Main Theorem (1), we may prove the following:

Lemma (3.1).

(1) Δ is not of types $D_{13} \oplus D_6$, $D_{10} \oplus D_9$, $D_4 \oplus D_{15}$.

Proof of (1). We shall argue by contradiction. Since $(S, g) \simeq (S_3, g_3)$ by (2.2), we may identify (S, g) with (S_3, g_3). We denote $\text{Supp}\Delta = (\cup_{i=1}^{3l+1} C_i) \coprod (\cup_{j=1}^{3m} E_j)$ where the numberings are given as $C_1.C_3 = C_2.C_3 = C_i.C_{i+1} = 1 (i \geq 3)$ and $E_1.E_3 = E_2.E_3 = E_j.E_{j+1} = 1 (j \geq 3)$. By Δ, we also denote the sublattice of $\text{Pic}S_3$ generated by the irreducible components of Δ. Let us consider the primitive closure $\overline{\Delta}$ of Δ in $\text{Pic}S_3$. Since $[\overline{\Delta} : \Delta]^2 = (\text{discr}\Delta)/(\text{discr}\overline{\Delta}) = 16/(\text{discr}\overline{\Delta})$, $[\overline{\Delta} : \Delta]$ is either 1, 2 or 4. Dividing into these three cases, we shall derive a contradiction.

First assume that $[\overline{\Delta} : \Delta] = 4$. Then $\text{discr}\overline{\Delta} = 1$. Thus, we have an othorgonal decomposition of $\text{Pic}S_3$: $\text{Pic}S_3 = \overline{\Delta} \oplus \mathbb{Z} \cdot H$. This implies $H^2 = \text{discr Pic}S_3 = 3 \not\equiv 0 (\text{mod}2)$, a contradiction.

Next assume that $[\overline{\Delta} : \Delta] = 2$. Then there exist integers $\alpha_i, \beta_j \in \{0, 1\}$ such that $L := \frac{1}{2}(\sum_{i=1}^{3l+1} \alpha_i C_i + \sum_{j=1}^{3m} \beta_j E_j) \in \overline{\Delta} - \Delta$. Substituting $i = 3l+1, 3l, ..., 3$ and $j = 3m, 3m-1, ..., 3$ into $L.C_i \in \mathbb{Z}$ and $L.E_j \in \mathbb{Z}$, we readily find that

$L = \frac{1}{2}(E_6 + E_4 + E_2)$ or $\frac{1}{2}(E_6 + E_4 + E_2)$ in the case where Δ is of type $D_{13} \oplus D_6$,

$L = \frac{1}{2}(C_{10} + C_8 + C_6 + C_4 + C_2)$ or $\frac{1}{2}(C_{10} + C_8 + C_6 + C_4 + C_1)$ in the case where Δ is of type $D_{10} \oplus D_9$, and

$L = \frac{1}{2}(E_4 + E_2)$ or $\frac{1}{2}(E_4 + E_1)$ in the case where Δ is of type $D_4 \oplus D_{15}$.

But this contradicts the next Lemma due to Nikulin [N]:

Lemma (3.2). *Let $C_1, C_2, ..., C_l$ be mutually disjoint smooth rational curves on a smooth K3 surface T. Assume that $C_1 + \cdots + C_l \in 2 \cdot \text{Pic}T$. Then, l is either 0, 8, or 16.* $\square$

Finally assume that $[\overline{\Delta} : \Delta] = 1$. That is, Δ is primitive in $\text{Pic}S_3$. Then there exists an element $h \in \text{Pic}S_3$ such that $\text{Pic}S_3 = \mathbb{Z} < C_1, ..., C_{3l+1}, E_1, ..., E_{3m}, h >$. Set $\mathbb{Z} \cdot H = \Delta^{\perp}$ in $\text{Pic}S_3$ and $n = [\text{Pic}S_3 : \Delta \oplus \mathbb{Z} \cdot H]$. Then $n^2 = (\text{discr}\Delta \cdot H^2)/(\text{discr Pic}S_3) = 16H^2/3$, that is, $H^2 = \frac{3}{16}n^2$. On the other hand, by replacing h by $-h$ if necessary, we can find integers a_i, b_j such that $H = nh + \sum a_i C_i + \sum b_j E_j$ in $\text{Pic}S_3$, that is, $\frac{H}{n} = h + \sum \frac{a_i}{n} C_i + \sum \frac{b_j}{n} E_j$ in $\text{Pic}S_3 \otimes \mathbb{Q}$. Using $H.C_\alpha = H.E_\beta = 0$ and the negative definiteness of $(C_i \cdot C_\alpha)$ and $(E_j \cdot E_\beta)$, we see that $(\frac{a_i}{n}, \frac{b_j}{n})$ is the unique solution of

$$(h + \sum x_i C_i + \sum y_j E_j) \cdot C_\alpha = 0, (h + \sum x_i C_i + \sum y_j E_j) \cdot E_\beta = 0,$$

$(\alpha = 1, ..., 3l+1; \beta = 1, ..., 3m)$.

Since $\text{discr}(C_i \cdot C_\alpha) = \text{discr}(E_j \cdot E_\beta) = 4$, this implies that $\frac{a_i}{n}, \frac{b_j}{n} \in \frac{\mathbb{Z}}{4}$. Thus, $\frac{4H}{n} = 4h + \sum \frac{4a_i}{n} C_i + \sum \frac{4b_j}{n} E_j \in \text{Pic}S_3$. This implies $n|4$. However, then $H^2 = \frac{3}{16}n^2 \not\equiv 0 (\text{mod}2)$, a contradiction. This proves the assertion (1). $\square$

Proof of (2). The verification of (2) is quite similar to that of (1). Assuming the contrary, we identify (S, g) with (S_3, g_3) and set $\text{Supp}\Delta = (\cup_{i=1}^{10} C_i) \coprod (\cup_{j=1}^{9} E_j)$ where the names are given as: $\sum C_i$ is of type D_{10}, $C_1.C_3 = C_2.C_3 = C_i.C_{i+1} = 1 (i \geq 3)$, $\sum E_j$ is of type A_9 and $E_j.E_{j+1} = 1 (j \geq 1)$. Let $\overline{\Delta}$ be the primitive closure of the sublattice Δ in $\text{Pic}S_3$. Again, it follows from $[\overline{\Delta} : \Delta]^2 = (\text{discr}\Delta)/(\text{discr}\overline{\Delta}) = 40/(\text{discr}\overline{\Delta})$ that $[\overline{\Delta} : \Delta]$ is either 1 or 2. In each case, we shall derive a contradiction.

First assume that $[\overline{\Delta} : \Delta] = 2$. Then there exists integers $\alpha_i, \beta_j \in \{0, 1\}$ such

is either one of $\frac{\delta_1}{2}(C_{10} + C_8 + C_6 + C_4 + C_i) + \frac{\delta_2}{2}(E_9 + E_7 + E_5 + E_3 + E_1)$, where $i \in \{1, 2\}$, $(\delta_1, \delta_2) \in \{(0,1), (1,0), (1,1)\}$. However this is against (3.2).

Next assume that $[\overline{\Delta} : \Delta] = 1$, that is, Δ is primitive in $\mathrm{Pic}S_3$. Then, as before, there exists an element $h \in \mathrm{Pic}S_3$ such that $\mathrm{Pic}S_3 = \Delta + \mathbb{Z} \cdot h$. Set $\mathbb{Z} \cdot H = \Delta^{\perp}$ in $\mathrm{Pic}S_3$ and $n = [\mathrm{Pic}S_3 : \Delta \oplus \mathbb{Z} \cdot H]$. Then $n^2 = ((\mathrm{discr}\Delta) \cdot H^2)/(\mathrm{discr}\ \mathrm{Pic}S_3) = 40H^2/3$ and (by replacing h by $-h$ if necessary,) we can find integers a_i, b_j such that $H = nh + \sum a_i C_i + \sum b_j E_j$ in $\mathrm{Pic}S_3$, that is, $\frac{H}{n} = h + \sum \frac{a_i}{n} C_i + \sum \frac{b_j}{n} E_j$ in $\mathrm{Pic}S_3 \otimes \mathbb{Q}$. Using $H.C_\alpha = H.E_\beta = 0$ and the negative definiteness of $(C_i \cdot C_\alpha)$ and $(E_j \cdot E_\beta)$, we see that $(\frac{a_i}{n}, \frac{b_j}{n})$ is the unique solution of

$$(h + \sum x_i C_i + \sum y_j E_j) \cdot C_\alpha = 0, (h + \sum x_i C_i + \sum y_j E_j) \cdot E_\beta = 0,$$

$(\alpha = 1, ..., 10; \beta = 1, ..., 9)$.

Since $\mathrm{discr}(C_i \cdot C_\alpha) = 4$ and $\mathrm{discr}(E_j \cdot E_\beta) = 10$, this implies that $\frac{a_i}{n} \in \frac{\mathbb{Z}}{4}$ and $\frac{b_j}{n} \in \frac{\mathbb{Z}}{10}$. Thus, $\frac{20H}{n} = 20h + \sum \frac{20a_i}{n} C_i + \sum \frac{20b_j}{n} E_j \in \mathrm{Pic}S_3$ and in particular, $n|20$. Combining this with $H^2 = 3n^2/40 \equiv 0 \pmod{2}$, we find that $n = 20$ and $H^2 = 30$. Thus, $H = 20h + \sum a_i C_i + \sum b_j E_j$, $a_i \in 5 \cdot \mathbb{Z}$ and $b_j \in 2 \cdot \mathbb{Z}$. Set $a_i = 5\alpha_i$ and $b_j = 2\beta_j$. Then,

(1) $H = 20h + 5(\sum \alpha_i C_i) + 2(\sum \beta_j E_j)$, and
(2) $400h^2 = (20h)^2 = H^2 + 25(\sum \alpha_i C_i)^2 + 4(\sum \beta_j E_j)^2$,
(3) $0 = H \cdot (\sum \alpha_i C_i) = 20(h \cdot \sum \alpha_i C_i) + 5(\sum \alpha_i C_i)^2$, and
(4) $0 = H \cdot (\sum \beta_j E_j) = 20(h \cdot \sum \beta_j E_j) + 2(\sum \beta_j E_j)^2$.

By (3) and (4), we see that $5(\sum \alpha_i C_i)^2) \equiv 2(\sum \beta_j E_j)^2) \equiv 0 \pmod{20}$. Substituting this into (2), we get $H^2 \equiv 0 \pmod{20}$. However this contradicts the previous equality $H^2 = 30$. Now we are done. $\square$

§4. CLASSIFICATION OF EXTREMAL LOG ENRIQUES SURFACES

In this section, we prove the Main Theorem (3). Throughout this section, we again work in the setting (2.1). By the Main Theorem (1), we know that Δ is now one of either A_{19}, D_{19}, $D_{16} \oplus A_3$, $D_{13} \oplus A_6$, $D_7 \oplus A_{12}$, $D_4 \oplus A_{15}$, or $D_7 \oplus D_{12}$.

In the case where Δ is either of type A_{19} or of type D_{19}, the result follows from [OZ1, Theorems 1 and 2]. So we may consider the remaining cases:

Case 1. $D_{16} \oplus A_3$, *Case* 2. $D_{13} \oplus A_6$, *Case* 3. $D_7 \oplus A_{12}$, *Case* 4. $D_4 \oplus A_{15}$, and *Case* 5. $D_7 \oplus D_{12}$.

Since in each case $(S, g) \simeq (S_3, g_3)$ ((2.1)), we identify these two in the sequel. Set $\Delta = (\cup_{i=1}^{3l+1} C_i) \coprod (\cup_{j=1}^{3m} E_j)$ where in Cases (1) - (5), $\sum C_i$ is of type D_{3l+1}, $C_{3l+1}.C_{3l-1} = C_{3l}.C_{3l-1} = C_i.C_{i+1} = 1(i \leq 3l-1)$, $\sum E_j$ is of type A_{3m} and $E_j.E_{j+1} = 1(j \geq 1)$, and in Case (5), $\sum C_i$ is of type D_7, $C_7.C_5 = C_6.C_5 = C_j.C_{j+1} = 1(j \leq 4)$, $\sum E_j$ is of type D_{12} and $E_{12}.E_{10} = E_{11}.E_{10} = E_j.E_{j+1} = 1(j \leq 9)$. We also denote by the same letter Δ the sublattice of $\mathrm{Pic}S_3$ generated by the irreducible components of Δ and by $\overline{\Delta}$ its primitive closure in $\mathrm{Pic}S_3$ as in Section 3. Set $\mathbb{Z} \cdot H = \overline{\Delta}^{\perp} = \Delta^{\perp}$ in $\mathrm{Pic}S_3$. Here we may take H as the pull back of the ample generator of $\mathrm{Pic}\overline{S}$. For convenience of notation, we sometimes set $G_1 = C_1, ..., G_{3l+1} = C_{3l+1}, G_{3l+2} = E_1, ..., G_{19} = E_{3m}$.

Proposition (4.1). *In each case, Z is unique up to isomorphisms if the following two conditions are satisfied:*

(1) H^2 *is determined only by the type of Z and is independent of particular*

(2) *The dual graph of the divisor Δ determines, uniquely up to graph isomorphisms, rational numbers a_{ij}, b, b_k ($1 \leq i, j, k \leq 19$), such that $e_j := \sum_{i=1}^{19} a_{ij} G_i$ ($1 \leq j \leq 19$) form a $\mathbb{Z}-$basis of $\overline{\Delta}$ and e_j's, $e_{20} := bH + \sum_{k=1}^{19} b_k D_k$ form a $\mathbb{Z}-$basis of $\mathrm{Pic} S_3$.*

Proof. Let Z be an extremal log Enriques surface with Δ, G_i, H as defined above or (2.1). Let $Z(\alpha)$ be the extremal log Enriques surface in (1.7) of the same type as that of Z. As for Z, we can define similarly $\Delta(\alpha)$, $G_i(\alpha)$, $H(\alpha)$, etc. Then, by the conditions (1) and (2), there exists an isometry $\psi : \mathrm{Pic} S_3 \to \mathrm{Pic} S_3$ such that $\psi(D_i) = D_i(\alpha)$, $\psi(H) = H(\alpha)$ and that ψ preserves ample classes. The last condition follows from the fact, which is derived from Kleiman's criterion on ampleness, that there are sufficiently small positive numbers γ_k such that both $H - \sum \gamma_k D_k$ and $H(\alpha) - \sum \gamma_k D_k(\alpha)$ are ample divisors. Then by [V, page 13], ψ extends to an effective Hodge isometry $\overline{\psi}$ of $H^2(S_3, \mathbb{Z})$. Now we may apply the Torelli Theorem for K3 surfaces to get an automorphism φ of S_3 such that $\varphi^* = \overline{\psi}$. By construction, φ maps the exceptional divisor Δ to $\Delta(\alpha)$. Combining this with the result $g_3 \circ \varphi = \varphi \circ g_3$ in (1.2), we see that φ is an equivariant isomorphism between the triplets (S_3, g_3, Δ) and $(S_3, g_3, \Delta(\alpha))$. Thus φ descends to an isomorphism $Z \to Z(\alpha)$. $\square$

Now we may check the conditions (1) and (2) for each case.
Case 1, the case where Δ is of type $D_{16} \oplus A_3$

Claim (4.2).

(1) $[\overline{\Delta} : \Delta] = 2$.
(2) *Up to $\mathrm{Aut}_{graph}(D_{16})$, $e_1 := \frac{1}{2}(C_1 + C_3 + C_5 + \cdots + C_{13} + C_{16})$, $e_i = C_{17-i}$ ($2 \leq i \leq 16$), $e_{17} := E_1$, $e_{18} := E_2$, $e_{19} := E_3$ form a $\mathbb{Z}-$basis of $\overline{\Delta}$.*

Proof. Since $[\overline{\Delta} : \Delta]^2 = (\mathrm{discr}\Delta)/(\mathrm{discr}\overline{\Delta}) = 16/(\mathrm{discr}\overline{\Delta})$, $[\overline{\Delta} : \Delta]$ is either 1, 2 or 4.

If $[\overline{\Delta} : \Delta] = 1$ or 4, we will get a contradiction as in Section 3, noting that we also have $\mathrm{discr}(C_i \cdot C_\alpha) = \mathrm{discr}(E_j \cdot E_\beta) = 4$ here. This proves (1) of (4.2).

The fact that $[\overline{\Delta} : \Delta] = 2$ and the argument in (3.1) imply that $\overline{\Delta} - \Delta$ contains, after interchanging C_1 and C_2 by the non-trivial element in $\mathrm{Aut}_{graph}(D_{16})$ if necessary, either $L = \frac{1}{2}(C_1 + C_3 + \cdots + C_{13} + C_{16})$, or $L = \frac{1}{2}(E_1 + E_3)$, or $L = \frac{1}{2}(C_1 + C_3 + \cdots + C_{13} + C_{16}) + \frac{1}{2}(E_1 + E_3)$. Combining this with Nikulin's result (cf. Lemma (3.2)), we get $L = \frac{1}{2}(C_1 + C_3 + \cdots + C_{13} + C_{16}) + \frac{1}{2}(E_1 + E_3)$. This implies the assertion (2). $\square$

Claim (4.3).

(1) $H^2 = 12$.
(2) *Up to $\mathrm{Aut}_{graph}(\Delta)$, e_1, e_2, ..., e_{19} and $\frac{1}{4}(H - E_1 - 2E_2 - 3E_3)$ form a $\mathbb{Z}-$basis of $\mathrm{Pic} S_3$.*

Set $n = [\mathrm{Pic} S_3 : \overline{\Delta} \oplus \mathbb{Z} \cdot H]$ and $\mathrm{Pic} S_3 = \overline{\Delta} + \mathbb{Z} \cdot h$. Then, we have $n^2 = (\mathrm{discr}\overline{\Delta} \cdot H^2)/(\mathrm{discr\ Pic} S_3) = 4H^2/3$, that is, $H^2 = 3n^2/4$ and replacing h by $-h$ if necessary we can write $H = nh + \sum \alpha_k e_k$ (for some integers α_k). Using $\mathrm{discr}(e_i \cdot e_\alpha)_{i,\alpha=1,...,16} = 1$ and $\mathrm{discr}(e_i \cdot e_\alpha)_{i,\alpha=17,...,19} = 4$, we see that $\frac{\alpha_i}{n} \in \mathbb{Z}$ for $i = 1, ..., 16$ and $\frac{\alpha_i}{n} \in \frac{\mathbb{Z}}{4}$ for $i = 17, ..., 19$. In particular, $\frac{4H}{n} = 4h + \sum \frac{4\alpha_k}{n} e_k \in \mathrm{Pic} S_3$.

$H^2 = 12$. Thus, there exist integers c_i ($i = 1, ..., 16$), d_j ($j = 17, ..., 19$) such that $H = 4h + \sum_{i=1}^{16} 4c_i e_i + \sum_{j=17}^{19} d_j e_j$. Replacing h by $h - \sum m_k e_k$ ($m_k \in \mathbb{Z}$), and using $e_{j+16} = E_j$ if $j = 1, ..., 3$, we can adjust h like $h = \frac{1}{4}H - \frac{1}{4}\sum_{j=1}^{3} a_j E_j$ for some integers $a_j \in \{0, 1, 2, 3\}$. We shall determine a_j up to $\text{Aut}_{\text{graph}}(A_3)$. Using $h \cdot E_j \in \mathbb{Z}$, we get $-2a_1 + a_2 \equiv 0 (\text{mod} 4)$, $a_1 - 2a_2 + a_3 \equiv 0 (\text{mod} 4)$, and $a_2 - 2a_3 \equiv 0 (\text{mod} 4)$. Thus, up to $\text{Aut}_{\text{graph}}(A_3)$, we have either (1) $a_1 = a_2 = a_3 = 0$, or (2) $a_1 = a_3 = 2$ and $a_2 = 0$ or (3) $a_1 = 1$, $a_2 = 2$ and $a_3 = 3$.

In case (1), we calculate $h^2 = (\frac{1}{4}H)^2 = \frac{3}{4} \notin \mathbb{Z}$, a contradiction. Also, in case (2), we calculate $h^2 = (\frac{1}{4}H)^2 + \frac{1}{4}(E_1 + E_3)^2 = \frac{3}{4} - 1 \notin \mathbb{Z}$, a contradiction. Thus, the only possible values of a_j are $a_1 = 1$, $a_2 = 2$ and $a_3 = 3$. Since we already know the existence of such a_j, this gives the assertion (2). $\square$

Remark. It does not seem easy at least for the authors to find directly for which a_j, $\frac{1}{4}H - \frac{1}{4}\sum_{j=1}^{3} a_j E_j$ is really in $\text{Pic}S_3$. This is the reason why we argued as above.

Case 2, the case where Δ is of type $D_{13} \oplus A_6$

Claim (4.4). *The sublattice Δ is primitive in $\text{Pic}S_3$, i.e., Δ is equal to its primitive closure $\overline{\Delta}$ in $\text{Pic}S_3$.*

Proof. Since $\text{discr}\Delta = 4 \cdot 7$, if (4.4) is false we have $[\overline{\Delta} : \Delta] = 2$. Then we will reach a contradiction to (3.2) as in the proof of (3.1). $\square$

Claim (4.5).

(1) $H^2 = 84$ *and*

(2) *Up to $\text{Aut}_{\text{graph}}(\Delta)$, $C_1, ..., C_{12}, E_1, ..., E_6$ and $\frac{1}{28}H - \frac{1}{4}(2C_1 + 2C_3 + \cdots + 2C_9 + 2C_{11} + C_{12} + 3C_{13}) - \frac{1}{7}(\sum_{j=1}^{6} jE_j)$ are $\mathbb{Z}-$basis of $\text{Pic}S_3$.*

Proof. Set $\text{Pic}S_3 = \Delta + \mathbb{Z} \cdot h$ and $n = [\text{Pic}S_3 : \Delta \oplus \mathbb{Z} \cdot H]$. Then by the same argument as before, we see that

$n^2 = (\text{dicsr}\Delta \cdot H^2)/(\text{discr Pic}S_3) = 28H^2/3$,

$H = nh + \sum_i \alpha_i C_i + \sum_j \beta_j E_j$, and

$\frac{\alpha_i}{n} \in \frac{\mathbb{Z}}{4}$, $\frac{\beta_j}{n} \in \frac{\mathbb{Z}}{7}$. Thus $\frac{28}{n}H = 28h + 7(\sum \frac{4\alpha_i}{n}C_i) + 4(\sum \frac{7\beta_j}{n}E_j) \in \text{Pic}S_3$ and then $n|28$. Combining this with $H^2 = \frac{3}{28}n^2 \equiv 0 (\text{mod} 2)$, we get $n = 28$ and $H^2 = 84$. Thus, replacing h by $h - \sum_i m_i C_i - \sum_j n_j E_j$ ($m_i, n_j \in \mathbb{Z}$), we may adjust h such as $h = \frac{1}{28}H - \frac{1}{4}(\sum_i a_i C_i) - \frac{1}{7}(\sum_j b_j E_j)$ where $a_i \in \{0, 1, 2, 3\}$ and $b_j \in \{0, 1, ..., 6\}$. We determine a_i and b_j up to $\text{Aut}_{\text{graph}}(\Delta)$. By $\text{Aut}_{\text{graph}}(\Delta)$, we may assume $a_{12} \leq a_{13}$ and $b_1 \leq b_6$. Using $h \cdot C_k \in \mathbb{Z}$, we can readily see that $a_i \equiv ia_i (\text{mod} 4)$ for $1 \leq i \leq 11$, $a_{10} + a_{12} + a_{13} - 2a_{11} \equiv 0 (\text{mod} 4)$, $-2a_{12} + a_{11} \equiv 0 (\text{mod} 4)$, and $-2a_{13} + a_{11} \equiv 0 (\text{mod} 4)$. These formulas imply that a_i are either,

(1) $a_1 = a_2 = \cdots = a_{13} = 0$,

(2) $a_1 = a_2 = \cdots = a_{11} = 0$, $a_{12} = a_{13} = 2$, or

(3) $a_1 = a_3 = \cdots = a_9 = a_{11} = 2$, $a_2 = a_4 = \cdots = a_{10} = 0$, $a_{12} = 1$ and $a_{13} = 3$.

Using $h \cdot E_l \in \mathbb{Z}$, we have $b_j \equiv jb_1 (\text{mod} 7)$ and b_1 is either $0, 1, 2, 3$. The assertion here follows from our assumption that $b_1 \leq b_6$. Thus, according to (1), (2), (3), $h^2 \equiv \frac{3}{28} + \frac{b_1^2}{7} (\text{mod}\mathbb{Z})$, $\equiv \frac{3}{28} - 1 + \frac{b_1^2}{7} (\text{mod}\mathbb{Z})$, and $\equiv \frac{3}{28} - \frac{13}{4} + \frac{b_1^2}{7} (\text{mod}\mathbb{Z})$. Thus, by $h^2 \equiv 0 (\text{mod} 2 \cdot \mathbb{Z})$, we see that $b_1 = 1$ and that a_i satisfy (3). This proves (4.5). $\square$

By the same argument as in case 2, we get the following two claims, which guarantee the conditions (1) and (2) in (4.1).

Claim (4.6). Δ *is primitive in* $PicS_3$.

Claim (4.7).
1. $H^2 = 156$ *and*
2. *Up to* $Aut_{graph}(\Delta)$, $C_1, ..., C_7, E_1, ..., E_{12}$ *and* $\frac{1}{52}H - \frac{1}{4}(2C_1 + 2C_3 + 2C_5 + C_6 + 3C_7) - \frac{1}{7}(\sum_{j=1}^{12} \overline{2j}E_j)$ *are* $\mathbb{Z}-$*basis of* $PicS_3$, *where we denote by* $\overline{2j}$ *the integer determined by* $2j \equiv \overline{2j}(mod13)$ *and* $0 \le \overline{2j} \le 12$.

Case 4, the case where Δ *is of type* $D_4 \oplus A_{15}$
This is the hardest case.

Claim (4.8).
1. $[\overline{\Delta} : \Delta] = 4$ *and* $\overline{\Delta}/\Delta \simeq \mathbb{Z}/4$.
2. $e_1 := C_1$, $e_2 := C_2$, $e_3 := C_3$, $e_4 := C_4$, $e_5 := E_2$, $e_6 := E_3$, $\cdots$ $e_{17} := E_{14}$, $e_{18} := E_{15}$ *and* $e_{19} := \frac{1}{2}(C_1 + C_2) + \frac{1}{4}(E_1 + 2E_2 - E_3 + E_5 + 2E_6 - E_7 + E_9 + 2E_{10} - E_{11} + E_{13} + 2E_{14} - E_{15})$ *are* $\mathbb{Z}-$*basis of* $\overline{\Delta}$.

Proof of (1). Since $\mathrm{discr}\Delta = 4 \cdot 16 = 64$, we have either (i) $[\overline{\Delta} : \Delta] = 8$, (ii) $[\overline{\Delta} : \Delta] = 1$, (iii) $[\overline{\Delta} : \Delta] = 2$, (iv) $[\overline{\Delta} : \Delta] = 4$ and $\overline{\Delta}/\Delta \simeq (\mathbb{Z}/2)^{\oplus 2}$, or (v) $[\overline{\Delta} : \Delta] = 4$ and $\overline{\Delta}/\Delta \simeq \mathbb{Z}/4$.

We elminate the cases (i) - (iv) by arguing by contradiction.

Case (i). In this case $\mathrm{discr}\overline{\Delta} = 1$. Then, $H^2 = \mathrm{discr}\ PicS_3 = 3 \not\equiv 0(mod2)$, a contradiction.

Case (ii). We have $\overline{\Delta} = \Delta$. Set $n = [PicS_3 : \Delta \oplus \mathbb{Z} \cdot H]$. Then, $n^2 = \frac{64}{3}H^2$ and $H = nh + \sum_i a_i C_i + \sum_j b_j E_j$ for some integers a_i, b_j. Since $\mathrm{discr}(C_i \cdot C_\alpha) = 4$ and $\mathrm{discr}(E_j \cdot E_\beta) = 16$, we see that $\frac{4a_i}{n}, \frac{16b_j}{n} \in \mathbb{Z}$ and that $\frac{16}{n}H = 16h + \sum \frac{16a_i}{n}C_i + \sum \frac{16b_j}{n}E_j \in PicS_3$. Thus $n|16$. Combining this with $H^2 = \frac{3}{64}n^2 \equiv 0(mod2)$, we get $n = 16$ and $H^2 = 12$. Then $H = 16h + \sum_i 4\alpha_i C_i + \sum_j b_j E_j$, where $\alpha_i = \frac{a_i}{4}(\in \mathbb{Z})$. Using this formula, we calculate $16^2 h^2 = H^2 + 16(\sum_i \alpha_i C_i)^2 + (\sum_j b_j E_j)^2$. On the other hand, since $0 = H \cdot (\sum_j b_j E_j) = 16h \cdot (\sum_j b_j E_j) + (\sum_j b_j E_j)^2$, we find $(\sum_j b_j E_j)^2 \equiv 0(mod16)$. Then $12 = H^2 \equiv 0(mod16)$, a contradiction.

Case (iii). In this case, there exist integers $\alpha_i, \beta_j \in \{0, 1\}$ such that $L := \frac{1}{2}(\sum_{i=1}^4 \alpha_i C_i + \sum_{j=1}^{15} \beta_j E_j) \in \overline{\Delta} - \Delta$. Since $L.C_i \in \mathbb{Z}$ and $L.E_j \in \mathbb{Z}$, we readily find that $L = \frac{1}{2}(E_1 + E_3 + \cdots + E_{13} + E_{15})$ and that $C_1, ..., C_4, G_1 := E_1, G_2 := E_2, ..., G_{14} := E_{14}$, and $G_{15} := L$ are $\mathbb{Z}-$basis of $\overline{\Delta}$. Set $n = [PicS_3 : \overline{\Delta} \oplus \mathbb{Z} \cdot H]$ and $PicS_3 = \overline{\Delta} + \mathbb{Z} \cdot h$. Then $n^2 = \frac{16}{3}H^2$ and $H = nh + \sum_i a_i C_i + \sum_j b_j G_j$ for some integers a_i, b_j. Since $\mathrm{discr}(C_i \cdot C_\alpha) = \mathrm{discr}(G_j \cdot G_\beta) = 4$, we have $\frac{a_i}{n}, \frac{b_j}{n} \in \frac{\mathbb{Z}}{4}$, that is, $\frac{4H}{n} = 4h + \sum_i \frac{4a_i}{n}C_i + \sum_j \frac{4b_j}{n}G_j \in PicS_3$. Thus $n|4$. Then $H^2 = \frac{3}{16}n^2 \not\equiv 0(mod2)$, a contradiction.

Case (iv). In this case there should exist at least two $L_1, L_2 \in \overline{\Delta} - \Delta$. However, the same argument as in case (3) shows that such L_i is unique, namely $\frac{1}{2}(E_1 + E_3 + \cdots + E_{13} + E_{15})$, a contradiction.

Now the assertion (1) is proved. $\square$

Proof of (2). By (1), there exist subsets $I \subset \{1, 2, 3, 4\}$, $J \subset \{1, 2, ..., 15\}$ and inte-

$\overline{\Delta} - \Delta$ and that $2N \notin \Delta$. We determine N. Set $I' := \{i \in I | \alpha_i \neq 2\}$, $J' := \{j \in J | \beta_j \neq 2\}$ and $N' := \sum_{i \in I'} |\alpha_i| C_i + \sum_{j \in J'} |\beta_j| E_j$. Then $0 \equiv 4N \equiv N' (\mathrm{mod} 2 \cdot \mathrm{Pic} S_3)$. On the other hand, since $2N \notin \Delta$, $I' \neq \phi$ or $J' \neq \phi$. Using $N' \cdot C_i \equiv N' \cdot E_j \equiv 0 (\mathrm{mod} 2)$ and (3.2), we find that $I' = \phi$ and $J' = \{1, 3, 5, ..., 13, 15\}$. Replacing N by $-N$ if necessary, we may assume that $\beta_1 = 1$. Set $M := E_1 + 2E_2 - E_3 + E_5 + 2E_6 - E_7 + E_9 + 2E_{10} - E_{11} + E_{13} + 2E_{14} - E_{15}$. Then using $N \cdot C_i \in \mathbb{Z}$ and $N \cdot E_j \in \mathbb{Z}$, we readily see that (up to $Aut_{graph}(\Delta)$,) N is either (1) $\frac{1}{2}(C_1 + C_2) + \frac{1}{4}M$ or (2) $\frac{1}{4}M$. However, in case (2), $N^2 = \frac{1}{16}M^2 = -3 \not\equiv 0 (\mathrm{mod} 2)$, a contradiction. Thus $N = \frac{1}{2}(C_1 + C_2) + \frac{1}{4}M = e_{19}$. This implies the assertion (2). $\square$

Claim (4.9).

(1) $H^2 = 12$.

(2) (up to $Aut_{graph}(\Delta)$,) e_1, e_2, ..., e_{19} and $e_{20} := \frac{1}{4}(H - e_1 - 3e_2 - 2e_4 - 2e_6 - 2e_7 - 2e_8 - 2e_9 - 2e_{14} - 2e_{15} - 2e_{16} - 2e_{17} - 2e_{19})$ are $\mathbb{Z}-$basis of $\mathrm{Pic} S_3$.

Proof. Set $n = [\mathrm{Pic} S_3 : \overline{\Delta} \oplus \mathbb{Z} \cdot H]$ and $\mathrm{Pic} S_3 = \overline{\Delta} + \mathbb{Z} \cdot h$. Then using the same argument as in case 1 based on $\mathrm{discr} \overline{\Delta} = 4$, we get $n = 4$, $H^2 = 12$ and find integers $a_k \in \{0, 1, 2, 3\}$ ($1 \leq k \leq 19$) such that $e_1, ..., e_{19}$ and $e := \frac{1}{4}(H - \sum_k a_k e_k)$ are $\mathbb{Z}-$ basis of $\mathrm{Pic} S_3$. We determine a_i up to $Aut_{graph}(\Delta)$. Since $(\sum_k a_k e_k) \cdot e_l = -4e \cdot e_l \equiv 0 (\mathrm{mod} 4)$, we see that a_k are either

(1) $a_{19} = a_{18} = \cdots a_5 = a_4 = \cdots a_1 = 0$,

(2) $a_{19} = a_{18} = \cdots a_5 = 0$, $a_4 = \cdots = a_1 = 2$, or

(3) $a_{19} = 2$, $a_{18} = 0$, $a_{17} = \cdots a_{14} = 2$, $a_{13} = \cdots a_{10} = 0$, $a_9 = \cdots = a_6 = 2$, $a_5 = 0$, $a_4 = 2$, $a_3 = 0$, $a_2 = 3$, and $a_1 = 1$ (up to $Aut_{graph}(D_4)$).

However, in cases of (1) and (2), we see that $e^2 \notin \mathbb{Z}$, a contradiction. Thus the case (3) occurs, that is, $e = e_{20}$. $\square$

Case 5, the case where Δ is of type $D_7 \oplus D_{12}$

The verification is also quite similar. We only indicate Claims needed to check the conditions (1) and (2) in (4.1).

Claim (4.10).

(1) $[\overline{\Delta} : \Delta] = 2$.

(2) (Up to $Aut_{graph}(\Delta)$) $e_1 := C_1, e_2 := C_2, ..., e_6 := C_6$, $e_7 := \frac{1}{2}(C_6 + C_7 + E_1 + E_3 + \cdots + E_9 + E_{11})$, $e_8 := E_1$, $e_9 := E_2$, ..., $e_{19} := E_{12}$ are $\mathbb{Z}-$basis of $\overline{\Delta}$.

Claim (4.11).

(1) $H^2 = 12$.

(2) Up to an element of $Aut_{graph}(\Delta)$ which keeps $e_1, ..., e_{19}$ invariant, $e_1, ..., e_{19}$ and $e_{20} := \frac{1}{4}(H - 2e_1 - 2e_2 - \cdots - 2e_7 - e_8 - e_{10} - \cdots - e_{16} + e_{18} - e_{19})$ are $\mathbb{Z}-$basis of $\mathrm{Pic} S_3$.

Now we have completed the proof of the Main Theorem (3). Q.E.D.

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